SCHUR–WEYL DUALITY AND IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{sl}_n$

BEN LIM

ABSTRACT. We introduce results on the irreducible representations of the symmetric group $\mathfrak{S}_d$ together with some examples. The Schur functor will be defined and we prove Schur–Weyl Duality. We use this theory to calculate the character of the Schur functor. We will then give an explicit description for the irreducible representations of $\mathfrak{sl}_n$.

CONTENTS

1. INTRODUCTION

We recall a result from highest weight theory of the complex Lie algebra $\mathfrak{sl}_n$. Namely, to each tuple of $n-1$ positive integers $(a_1, \ldots, a_{n-1})$, there is exactly one irreducible representation of highest weight $a_1L_1 + \ldots + a_{n-1}(L_1 + \ldots + L_{n-1})$. Though this result is exciting in its own right, its proof does not say explicitly what such a representation is. This final project is motivated by the idea of going a step further to describe explicitly this irreducible representation of $\mathfrak{sl}_n$ and calculate its dimension. The way we will approach this is by first establishing a certain duality between the representation theory of $\mathfrak{S}_d$ and that of $\mathrm{GL}_n(\mathbf{C})$. This will be made precise via Theorem 3.3 of Section 3. In some circles, this theorem (which has several statements) is called Schur–Weyl duality. It was Schur in his 1901 thesis that classified all the irreducible polynomial representations of $\mathrm{GL}_n(\mathbf{C})$, and Weyl who made the construction as presented in [FH91].

Our approach to Schur–Weyl duality and the irreducible representations of $\mathfrak{sl}_n$ is largely based on the treatment given in [FH91], specifically Chapters 6 and 15 respectively. There are other approaches to Schur–Weyl duality; for an abstract form concerning centralizers of certain algebras, we refer the reader to Chapter 9 of [Pro07]. For the combinatorially minded, we refer to [Ful97], specifically Chapter 8 that describes the Schur functor as the solution to a certain universal problem. Lastly, [Bum04] approaches Schur–Weyl duality via representation rings and Lie groups. Character calculations of the Schur functor here are done by taking advantage of the existence of a ring homomorphism between the representation ring of the symmetric group and the ring of symmetric polynomials.

Summary of Contents. The material in this project begins with Section 2 where we introduce Young diagrams and a theorem concerning the classification of irreducible representations of $\mathfrak{S}_d$. Section 3 concerns the bulk of this project. In here we will define the Schur functor and state the main result of Schur–Weyl duality in the form of Theorem 3.3. Section 4 will be devoted to calculating the character of the Schur functor and as a corollary its dimension. Finally, we will use this character calculation to show that given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of an integer d, the highest weight of the Schur functor $\mathbb{S}_\lambda(V)$ as an irreducible representation of $\mathfrak{sl}_n$ is $\lambda_1L_1 + \ldots \lambda_nL_n$. By setting $\lambda = (a_1 + \ldots + a_{n-1}, a_2 + \ldots + a_{n-1}, \ldots, a_{n-1}, 0)$ we get $\mathbb{S}_\lambda(V)$ as the irreducible representation of highest weight $a_1L_1 + \ldots + a_{n-1}(L_1 + \ldots + L_{n-1})$.

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2. Irreducible Representations of $\mathfrak{S}_d$

Recall that the number of irreducible representations of a finite group G up to isomorphism is the number of conjugacy classes. The symmetric group $\mathfrak{S}_d$ is special in that the number of conjugacy classes is in bijection with the number of partitions of the integer d. By a *partition* of an integer d we mean a tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with each λ_i a non-negative integer and with $\sum_{i=1}^n \lambda_i = d$. We also require that $\lambda_i \geq \lambda_{i+1}$ for all i. It is often convenient to allow one or more zeros to occur at the end; tuples that differ only by trailing zeros are then identified. We also define the *length* of the partition λ to be the largest i such that $\lambda_i \neq 0$. Hence if $\lambda = (\lambda_1, \ldots, \lambda_k)$ is some partition of d with $k \leq n$, there is no loss in generality in writing $\lambda = (\lambda_1, \ldots, \lambda_n)$ because we can put zeros in entries $k+1, \ldots, n$. Now to any partition λ of d is associated a *Young diagram*

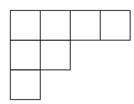


Figure 1. Young diagram corresponding to the partition $\lambda = (4, 2, 1)$ of 7.

with λ_i boxes in the i^{th} row. The upshot of considering Young diagrams is that we have a natural correspondence between the irreducible representations of $\mathfrak{S}_d$ and the Young diagrams of a partition of d. We may occasionally abuse notation and use λ to refer to both the partition and to the Young diagram. However this should be clear from the context. Now it would be useful if for a given Young diagram we could put a numbering on the boxes. We will call a numbering of the boxes with integers $1, 2, \ldots, d$ a *tableau* on the Young diagram. One way to number boxes is from left to right along a row, such as

1	2	3	4
5	6		
7			

Figure 2. Tableau on Young diagram for the partition $\lambda = (4, 2, 1)$ of 7.

We will often refer to the above numbering as the *standard tableau.* Putting the standard tableau on some Young diagram λ, we can define two subgroups of $\mathfrak{S}_d$, the *row group*

$$P_\lambda = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves every row}\}$$

and the *column group*

$$Q_\lambda = \{\tau \in \mathfrak{S}_d : \tau \text{ preserves every column}\}.$$

We can also define the associated sums a_λ and b_λ in the group algebra $\mathbf{C}[\mathfrak{S}_d]$ defined by

$$a_\lambda = \sum_{\sigma \in P_\lambda} e_\sigma, \quad b_\lambda = \sum_{\tau \in Q} \operatorname{sgn}(\tau) e_\tau.$$

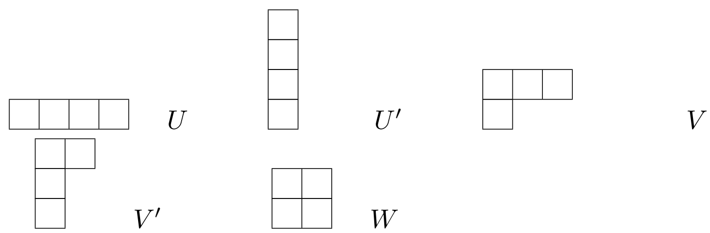
The group algebra $\mathbf{C}[\mathfrak{S}_d]$ is the complex vector space with basis elements indexed by $\mathfrak{S}_d$. Finally we define the quantity $c_\lambda = a_\lambda b_\lambda$, the *Young symmetrizer* associated to the partition λ. Let c_λ act on $\mathbf{C}[\mathfrak{S}_d]$ simply by right multiplication and consider its image $\mathbf{C}[\mathfrak{S}_d]c_\lambda$. Though we will not prove this here, the amazing fact about irreducible representations of $\mathfrak{S}_d$ is that they can be described completely by the following theorem:

Theorem 2.1. *Some scalar multiple of c_λ is idempotent, i.e. $c_\lambda^2 = n_\lambda c_\lambda$, and the image of c_λ (by right multiplication on $\mathbf{C}[\mathfrak{S}_d]$) is an irreducible representation V_λ of $\mathfrak{S}_d$. Every irreducible representation of $\mathfrak{S}_d$ can be obtained in this way for a unique partition.*

For a proof of this theorem, we refer the reader to Chapter 4 of [FH91]. Having discussed Young diagrams, we can proceed to examples that illustrate the correspondence between an irreducible representation and its associated Young diagram. As a first example consider $\mathfrak{S}_4$.

Examples 2.2.

There are 5 isomorphism classes of irreducible representations of $\mathfrak{S}_4$: They are the trivial representation U, the alternating representation U', the standard representation V that is 3 dimensional, the representation W lifted from the standard representation of $\mathfrak{S}_3$ and finally the tensor product $V \otimes U'$. In each case, it is not hard to work out the corresponding Young diagram:



Certainly for each d it is clear that the Young diagrams of the trivial and alternating representation follow a similar pattern. For $d \geq 5$ it may not be the case that we have the irreducible representations V' and W, but certainly we have the standard representation of $\mathfrak{S}_d$ that is $d-1$ dimensional. Those interested in a proof of its irreducibility are referred to [Ser97]. We may ask if the Young diagram of the standard representation for any d follows a pattern like that in the case $d = 4$. Indeed it does and we formulate it in the following proposition:

Proposition 2.3. *The Young diagram corresponding to the partition $\lambda = (d-1, 1)$ of d is that of the standard representation of $\mathfrak{S}_d$.*

Proof. We let a_λ act first on $\mathbf{C}[\mathfrak{S}_d]$, followed by b_λ. Now we claim that $\mathbf{C}[S_d]a_\lambda$ is d dimensional. To see this first notice that that the row group P_λ is isomorphic to $\mathfrak{S}_{d-1}$. From this it follows that given any e_σ, e_τ with $\sigma, \tau \in \mathfrak{S}_{d-1}$, we have $e_\sigma a_\lambda = e_\tau a_\lambda$. This is because a_λ is the sum of all elements in $\mathfrak{S}_{d-1}$ and multiplying again by an e_σ for $\sigma \in S_{d-1}$ just permutes the order of summation in a_λ. More generally, we see for any $e_\sigma, e_\tau \in \mathbf{C}[S_d]$ such that $\sigma^{-1}\tau \in \mathfrak{S}_{d-1}$ we have

$$e_\sigma a_\lambda = e_\tau a_\lambda.$$

This comes down to the fact that two left cosets $\sigma\mathfrak{S}_{d-1}$ and $\tau\mathfrak{S}_{d-1}$ are equal if and only if $\sigma^{-1}\tau \in \mathfrak{S}_{d-1}$. Now partition $\mathfrak{S}_d$ into left cosets $\rho\mathfrak{S}_{d-1}$ for ρ a 2-cycle of the form $(k\ d)$ for $1 \leq k \leq d$, with the convention that $(d\ d)$ is the identity. Then by the observation above for when two $e_\sigma a_\lambda$ and $e_\tau a_\lambda$ are equal, we have that $\mathbf{C}[\mathfrak{S}_d]a_\lambda$ has basis vectors v_i defined by

$$v_i = e_\rho a_\lambda$$

where ρ runs over all the coset representatives that we chose above. This completes the claim that $\mathbf{C}[\mathfrak{S}_d]$ is d-dimensional. We now apply b_λ to each of the basis vectors of $\mathbf{C}[\mathfrak{S}_d]a_\lambda$ and take their sum. A direct computation shows that this sum is zero, from which it follows that $\mathbf{C}[\mathfrak{S}_d]c_\lambda$ is $d-1$ dimensional with basis $v_2 b_\lambda,\ v_3 b_\lambda,\ \ldots, v_d b_\lambda$.

□

3. $\mathrm{GL}_n(\mathbf{C})$-$\mathfrak{S}_d$ Duality

3.1. **The Schur Functor.** Let V be a complex vector space of dimension n. Given any positive integer d we can consider the d-th *tensor power* $V^{\otimes d}$. This is simply the tensor product of d copies of V. The general linear group $\mathrm{GL}_n(\mathbf{C})$ acts on $V^{\otimes d}$ diagonally by defining

$$g \cdot (v_1 \otimes \ldots \otimes v_d) = gv_1 \otimes gv_2 \otimes \ldots gv_d$$

for any $g \in \mathrm{GL}_n(\mathbf{C})$ and vectors $v_1, \ldots, v_d \in V$. By embedding $\mathrm{GL}_n(\mathbf{C})$ in $\mathrm{End}(V^{\otimes d})$ we obtain a representation of $\mathrm{GL}_n(\mathbf{C})$. On the other hand, we have a right action of $\mathfrak{S}_d$ on $V^{\otimes d}$ given by

$$(v_1 \otimes \ldots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$$

for $\sigma \in \mathfrak{S}_d$. It may be checked with this definition that for any $\tau, \sigma \in \mathfrak{S}_d$ we have

$$((v_1 \otimes \dots v_d) \cdot \sigma) \cdot \tau = (v_1 \otimes \dots v_d) \cdot (\sigma\tau)$$

and that the left and right actions of $\mathrm{GL}_n(\mathbf{C})$ and $\mathfrak{S}_d$ commute. Extending linearly the right action of $\mathfrak{S}_d$ on $V^{\otimes d}$ we see that it becomes a $\mathrm{GL}_n(\mathbf{C}) - \mathbf{C}[\mathfrak{S}_d]$ bimodule. This is the basic idea of Schur–Weyl duality; to use the commuting actions $\mathrm{GL}_n(\mathbf{C})$ and $\mathfrak{S}_d$ on $V^{\otimes d}$ to relate representations of $\mathfrak{S}_d$ to those of $\mathrm{GL}_n(\mathbf{C})$. We are now ready to define the primary object of study in Schur–Weyl duality, the *Schur functor* or *Weyl module*:

Definition 3.1. *For any partition λ of d consider the Young symmetrizer c_λ defined in Section 2 as an endomorphism of $V^{\otimes d}$. We denote the image of c_λ on $V^{\otimes d}$ by $\mathbb{S}_\lambda(V)$. The functor $\mathbb{S}_\lambda$ that assigns to each V the space $\mathbb{S}_\lambda(V)$ is called the Schur functor.*

We justify the use of the word functor in the definition above. Consider first the functor $F : \mathbf{Vect} \to \mathbf{Vect}$ that assigns to each object $V \in \mathbf{Vect}$ the object $V^{\otimes d}$ and to each linear map $T : V \to W$ of $\mathbf{Vect}$ the linear map $T^{\otimes^d} : V^{\otimes d} \to W^{\otimes d}$. The action of $T^{\otimes d}$ on $V^{\otimes d}$ is diagonal, defined by $T^{\otimes d}(v_1 \otimes \dots v_d) = Tv_1 \otimes \dots Tv_d$. Because the action of $T^{\otimes d}$ commutes with that of c_λ, we can define $\mathbb{S}_\lambda(T) : \mathbb{S}_\lambda(V) \to \mathbb{S}_\lambda(W)$ to be the linear map that sends $v \cdot c_\lambda \in \mathbb{S}_\lambda(V)$ to $(T^{\otimes d}(v))c_\lambda$ for $v \in V^{\otimes d}$. If Id_V is the identity map on V, clearly $\mathbb{S}_\lambda(\mathrm{Id}_V)$ is the identity on $\mathbb{S}_\lambda(V)$. Also if $T : V \to W$ and $S : W \to U$ are linear maps, we have $\mathbb{S}_\lambda(S \circ T) = \mathbb{S}_\lambda(S) \circ \mathbb{S}_\lambda(T)$. Hence $\mathbb{S}_\lambda$ defines a functor from $\mathbf{Vect} \to \mathbf{Vect}$.

We note it may be for some partition λ that the Schur functor is actually the zero space. Thus we have the following proposition.

Definition 3.2. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of d. Then the Schur functor $\mathbb{S}_\lambda(V)$ is zero if and only if k is greater than $n = \dim V$.*

Proof. Suppose that λ has m columns of length $\mu_1, \mu_2, \dots, \mu_m$. The length of the first column is μ_1 and the length of the right most column is μ_m. Notice that $\mu_1 = k$, the number of rows in λ and that $\sum_{i=1}^m \mu_i = \sum_{j=1}^k \lambda_j$. Now suppose that $\mu_1 > n$. Then to show $\mathbb{S}_\lambda(V)$ is zero it suffices to show that $V^{\otimes d} b_\lambda = 0$. To do this we first decompose the column group Q_λ as a direct product of symmetric groups $\mathfrak{S}_{\mu_1} \times \dots \times \mathfrak{S}_{\mu_m}$. From this it follows that $b_\lambda = b_{\mu_1} b_{\mu_2} \dots b_{\mu_m}$. If we arrange the factors of $V^{\otimes d}$ as $V^{\otimes d} = V^{\otimes \mu_1} \otimes V^{\otimes \mu_2} \otimes \dots \otimes V^{\otimes \mu_m}$, we see that

$$\begin{aligned} V^{\otimes d} b_\lambda &= \left(V^{\otimes \mu_1} \otimes V^{\otimes \mu_2} \otimes \dots \otimes V^{\otimes \mu_m}\right) b_\lambda \\ &= V^{\otimes \mu_1} b_{\mu_1} \otimes V^{\otimes \mu_2} b_{\mu_2} \dots \otimes V^{\otimes \mu_m} b_{\mu_m} \\ &= \bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \dots \otimes \bigwedge^{\mu_m} V \end{aligned}$$

that is zero because $\dim \bigwedge^{\mu_1} V = \binom{\mu_1}{n} = 0$. For the converse, we refer the reader to Section 2, Chapter 9 of [Pro07].

□

3.2. **Schur–Weyl Duality.** Having defined the Schur functor in the previous Section, we are now ready to tie this together with the irreducible representations $V_\lambda = \mathbf{C}[\mathfrak{S}_d]c_\lambda$ of $\mathfrak{S}_d$ defined in Theorem 2.1.

Theorem 3.3 (Schur–Weyl Duality)**.** *Let V be an n-dimensional complex vector space on which $GL_n(\mathbf{C})$ acts via left multiplication.*

(1) If λ is a partition of d, the Schur functor $\mathbb{S}_\lambda(V)$ is isomorphic to $V^{\otimes d} \otimes_{\mathbf{C}[\mathfrak{S}_d]} V_\lambda$ as left $GL_n(\mathbf{C})$-modules.

(2) Let m_λ be the dimension of the irreducible representation V_λ of $\mathfrak{S}_d$ corresponding to a partition λ of d. Then as left $GL_n(\mathbf{C})$-modules, we have

$$V^{\otimes d} \cong \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\oplus m_\lambda}$$

where the direct sum taken over all partitions λ of d of at most n parts.

(3) Each $\mathbb{S}_\lambda(V)$ is an irreducible representation of $GL_n(\mathbf{C})$.

(4) The $GL_n(\mathbf{C})$-$\mathbf{C}[\mathfrak{S}_d]$ *bimodule* $V^{\otimes d}$ *is isomorphic to*

$$\bigoplus_{\lambda}(\mathbb{S}_\lambda(V)\otimes_{\mathbf{C}} V_\lambda)$$

where the direct sum is over all partitions of d *of at most* n *parts.*

Before we can prove Theorem 3.3, we will need the following lemma:

Lemma 3.4. *Let* U *be a finite dimensional right* A*-module.*

(i) For any $c \in A$*, the canonical map* $U \otimes_A Ac \to Ac$ *is an isomorphism of left* B*-modules.*
(ii) If $W = Ac$ *is an irreducible left* A*-module, then* $U \otimes_A W = Uc$ *is an irreducible left* B*-module.*
(iii) If $W_i = Ac_i$ *are the distinct irreducible left* A*-modules, with* m_i *the dimension of* W_i*, then*

$$U \cong \bigoplus_i (U\otimes_A W_i)^{\oplus m_i} \cong \bigoplus_i (Uc_i)^{\oplus m_i}$$

Proof.

(i) Consider the commutative diagram

$$\begin{array}{ccccc} U\otimes_A A & \xrightarrow{\cdot c} & U\otimes_A Ac & \xrightarrow{j} & U\otimes_A A \\ \downarrow & & \downarrow f & & \downarrow \\ U & \xrightarrow{\cdot c} & U\cdot c & \xrightarrow{i} & U \end{array}$$

with the maps i and j being inclusions; the vertical map f sends $v\otimes a \mapsto v\cdot a$. Now f is *a priori* only a group homomorphism. However because U is a $B-A$ bimodule, f also a left B-module homomorphism. Now the multiplication by c maps are clearly surjective while i and j are injective. U is clearly isomorphic to $U\otimes_A A$ from which it follows that f is an isomorphism of left B-modules.

(ii) We first assume that U is irreducible so that $B = \mathbf{C}$. In this case it will then suffice to prove that $U\otimes_A W \cong Uc$ is one dimensional or zero. First by Artin-Wedderburn we get that A is isomorphic to a direct sum of matrix rings $\bigoplus_{i=1}^r M_{n_i}(D_i)$ over some division ring D_i. Since there are no non-trivial finite dimensional division rings over $\mathbf{C}$, we conclude that $A = \bigoplus_{i=1}^r M_{n_i}(\mathbf{C})$. Now by assumption $W = Ac$ is an irreducible left A-module and hence is also a minimal left ideal of A. We will identify such a minimal ideal in a direct sum of matrix rings. Recall that an idempotent in a ring R is said to be primitive if it cannot be decomposed as the direct sum of two non-zero orthogonal idempotents.

In a semisimple ring such as $A = \bigoplus_{i=1}^r M_{n_i}(\mathbf{C})$, the primitive idempotents are hence those r-tuples of the form $(0,\dots,e,\dots,0)$ for e a primitive idempotent in $M_{n_i}(\mathbf{C})$ for some i. A primitive idempotent in $M_{n_i}(\mathbf{C})$ is just an $n_i\times n_i$ matrix E_{kk} for some $1\le k\le n_i$ with all entries zero except entry (k,k). By [Pro07, Theorem 3.1] every minimal left ideal in A is of the form $M_{n_i}(\mathbf{C})E_{kk}$ with E_{kk} a matrix of the form described above. Such a left ideal isomorphic to one that consists of tuples with all entries zero except entry i. In this entry, all matrices have only one non-zero column, namely column k. Similarly U can be identified with the right ideal of r-tuples which are zero except in factor j, and in that factor all are zero except row l say. It now follows that $U\otimes_A W$ will be zero unless $j = i$, in which case $U\otimes_A W$ is isomorphic to the set of matrices that are all zero except in entry (l,k). Hence $\dim U\otimes_A W \le 1$ and the proof in this case is complete. In general, decompose $U = \bigoplus_i U_i^{\oplus n_i}$ into a sum of irreducible right A-modules, so

$$U\otimes_A W \cong \bigoplus_i (U_i\otimes_A W)^{\oplus n_i} \cong (U_i\otimes_A W)^{n_k} \cong \mathbf{C}^{n_k}$$

for some k. This is clearly irreducible over $B = \bigoplus_j M_{n_j}(\mathbf{C})$.

(iii) If $W_i = Ac_i$ are the distinct irreducible left A-modules, with m_i the dimension of W_i then we can write $A = \bigoplus_i W_i^{\oplus m_i}$. Hence

$$U \cong U \otimes_A A \cong U \otimes_A \left(\bigoplus_i W_i^{\oplus m_i} \right) \cong \bigoplus_i (U \otimes_A W_i)^{\oplus m_i}.$$

By part (ii), each individual summand above (which is isomorphic to Uc_i by part (i)) is irreducible as a left B-module.

□

We are now ready to prove Theorem 3.3. The first will be devoted to proving Statements (1) to (3), the second exclusively to proving Statement (4).

Proof of Statements (1) to (3). We will first prove a more general result concerning semisimple rings in the form of Lemma 3.4 below, and then use this lemma for our specific purpose of Theorem 3.3. For the moment let G be any finite group, although our application is for the symmetric group. Now set $A = \mathbf{C}[G]$, the group algebra of G. By Mashcke's Theorem the group algebra $\mathbf{C}[G]$ is semisimple. Recall that a unital ring R (not necessarily commutative) is said to be semisimple if it is semisimple as a left module over itself. Now if U is any right A-module, let

$$B = \mathrm{Hom}_G(U,U) = \{\varphi : U \to U : \varphi(v \cdot g) = \varphi(v) \cdot g \;\; \forall v \in U, g \in G\}.$$

We note that B acts on U on the left, commuting with the right action of A; B is called the commutator algebra. Now recall that the direct sum of semisimple modules is semisimple, as is the quotient of a semisimple module. Since every module is a quotient of a free module, we get that U is semisimple as a right A-module. Hence if $U = \bigoplus U_i^{\oplus n_i}$ is an irreducible decomposition of U with U_i non-isomorphic right A-modules, by Schur's Lemma we have

$$B = \bigoplus_j \mathrm{Hom}_G(U_j^{\oplus n_j}, U_j^{\oplus n_j}) = \bigoplus_j M_{n_j}(\mathbf{C}),$$

where $M_{n_j}(\mathbf{C})$ is the ring of $n_j \times n_j$ complex matrices. If W is any left A-module, the tensor product

$$U \otimes_A W = U \otimes W / \text{subspace generated by } \{va \otimes w - v \otimes aw\}$$

is a left B-module by acting on the first factor, namely $b \cdot (v \otimes w) = (v \cdot v) \otimes w$.

Lemma 3.4 above tells us how to decompose U as a left B-module in the case that $U = V^{\otimes d}$, $B = \mathrm{Hom}_{\mathfrak{S}_d}(V^{\otimes d}, V^{\otimes d})$ and $A = \mathbf{C}[\mathfrak{S}_d]$. The question now is to pass from a decomposition of U as a left B-module to a left $\mathrm{GL}_n(\mathbf{C})$-module. The following lemma makes this possible.

Lemma 3.5. *The algebra B is spanned as a linear subspace of End$(V^{\otimes d})$ by End(V). A subspace of $V^{\otimes d}$ is a sub B-module if and only if it is invariant by $GL_n(\mathbf{C})$.*

Proof. [FH91, Lemma 6.23]. □

We notice now that $\mathbb{S}_\lambda(V)$ is Uc_λ, so Statement (1) follows from (i) of Lemma 3.4. Statement (3) follows from (ii) while Statement (2) from (iii). □

Proof of Statement (4). We prove first a general lemma concerning modules over the group algebra $\mathbf{C}[\mathfrak{S}_d]$.

Lemma 3.6. *Let U be a right A-module where $A = \mathbf{C}[\mathfrak{S}_d]$. If $U = \bigoplus U_i^{\oplus n_i}$ is a decomposition of U into irreducibles with U_i not isomorphic to U_j for $i \neq j$, then*

$$Hom_A(U_i, U) \otimes_{\mathbf{C}} U_i \cong U_i^{\oplus n_i}$$

via the map F that sends an elementary tensor $f \otimes v$ to $f(v)$.

Proof. The universal property of the tensor product guarantees that F is a well-defined group homomorphism that is also a homomorphism of right A-modules. Now *a priori* for each $v \in U$ and $f \in \mathrm{Hom}_A(U_i, U)$, $f(v)$

lands in U. However U_i irreducible implies that f is an isomorphism onto its image and so $f(v)$ lands in $U_i^{\oplus n_i}$. It is clear that the map F is surjective. Since the Hom functor commutes with direct sums, it follows

$$\mathrm{Hom}_A(U_i, U) \cong \mathrm{Hom}_A(U_i, U_i^{\oplus n_i}) \cong \bigoplus_{k=1}^{n_i} \mathrm{Hom}_A(U_i, U_i)_k$$

with each piece in the summand 1-dimensional by Schur's Lemma. It follows that $\dim \mathrm{Hom}_A(U_i, U) = n_i$ and the lemma is proven. □

We note that the abelian group $\mathrm{Hom}_A(U_i, U)$ *a priori* does not have the structure of a left A-module. It does however have the structure of a left $\mathrm{End}_A(U)$-module simply by function composition. However, if we apply Lemma 3.6 in the case that $U = A = \mathbf{C}[\mathfrak{S}_d]$ then

$$\mathrm{Hom}_A(U_i, U) \cong \mathrm{Hom}_{\mathbf{C}[\mathfrak{S}_d]}\left(c_\lambda \mathbf{C}[\mathfrak{S}_d], \mathbf{C}[\mathfrak{S}_d]\right) \tag{3.1}$$

has now the structure of a left $\mathbf{C}[\mathfrak{S}_d]$-module because we can identify left multiplication by some $a \in \mathbf{C}[\mathfrak{S}_d]$ with $g \in \mathrm{End}_{\mathbf{C}[\mathfrak{S}_d]}(\mathbf{C}[\mathfrak{S}_d])$ that maps x to ax. We have identified U_i with a minimal right ideal of $\mathbf{C}[\mathfrak{S}_d]$, i.e. $U_i = c_\lambda \mathbf{C}[\mathfrak{S}_d]$ for some λ a partition of d and c_λ the Young symmetrizer in isomorphism (3.1) above. Suppose for the moment that we consider $\mathbf{C}[\mathfrak{S}_d]$ tautologically as a $\mathbf{C}[\mathfrak{S}_d] - \mathbf{C}[\mathfrak{S}_d]$ bimodule. Then Lemma 3.6 tells us that

$$\mathbf{C}[\mathfrak{S}_d] \cong \bigoplus_\lambda \left(\mathrm{Hom}_{\mathbf{C}[\mathfrak{S}_d]}\left(c_\lambda \mathbf{C}[\mathfrak{S}_d], \mathbf{C}[\mathfrak{S}_d]\right) \otimes_{\mathbf{C}} c_\lambda \mathbf{C}[\mathfrak{S}_d]\right) \tag{3.2}$$

as $\mathbf{C}[\mathfrak{S}_d] - \mathbf{C}[\mathfrak{S}_d]$ bimodules. The sum is taken over all partitions λ of d. Now consider the $\mathrm{GL}_n(\mathbf{C}) - \mathbf{C}[\mathfrak{S}_d]$ isomorphism

$$V^{\otimes d} \cong V^{\otimes d} \otimes_{\mathbf{C}[\mathfrak{S}_d]} \mathbf{C}[\mathfrak{S}_d]$$

where $\mathrm{GL}_n(\mathbf{C})$ acts on the left factor of the tensor product, $\mathbf{C}[\mathfrak{S}_d]$ on the right factor. By the isomorphism in (3.2), we have

$$V^{\otimes d} \cong V^{\otimes d} \otimes_{\mathbf{C}[\mathfrak{S}_d]} \mathbf{C}[\mathfrak{S}_d] \tag{3.3}$$

$$\cong V^{\otimes d} \otimes_{\mathbf{C}[\mathfrak{S}_d]} \left(\bigoplus_\lambda \left(\mathrm{Hom}_{\mathbf{C}[\mathfrak{S}_d]}\left(c_\lambda \mathbf{C}[\mathfrak{S}_d], \mathbf{C}[\mathfrak{S}_d]\right) \otimes_{\mathbf{C}} c_\lambda \mathbf{C}[\mathfrak{S}_d]\right)\right) \tag{3.4}$$

$$\cong \bigoplus_\lambda \left(V^{\otimes d} \otimes_{\mathbf{C}[\mathfrak{S}_d]} \left(\mathrm{Hom}_{\mathbf{C}[\mathfrak{S}_d]}\left(c_\lambda \mathbf{C}[\mathfrak{S}_d], \mathbf{C}[\mathfrak{S}_d]\right) \otimes_{\mathbf{C}} c_\lambda \mathbf{C}[\mathfrak{S}_d]\right)\right) \tag{3.5}$$

$$\cong \bigoplus_\lambda \left(\left(V^{\otimes d} \otimes_{\mathbf{C}[\mathfrak{S}_d]} \mathrm{Hom}_{\mathbf{C}[\mathfrak{S}_d]}\left(c_\lambda \mathbf{C}[\mathfrak{S}_d], \mathbf{C}[\mathfrak{S}_d]\right)\right) \otimes_{\mathbf{C}} c_\lambda \mathbf{C}[\mathfrak{S}_d]\right) \tag{3.6}$$

as $\mathrm{GL}_n(\mathbf{C}) - \mathbf{C}[\mathfrak{S}_d]$ bimodules. Now

$$\mathrm{Hom}_{\mathbf{C}[\mathfrak{S}_d]}(c_\lambda \mathbf{C}[\mathfrak{S}_d], \mathbf{C}[\mathfrak{S}_d]) \cong \mathbf{C}[\mathfrak{S}_d] c_\lambda \tag{3.7}$$

via the isomorphism that sends $x \in \mathbf{C}[\mathfrak{S}_d] c_\lambda$ to the map $f_x : c_\lambda \mathbf{C}[\mathfrak{S}_d] \to \mathbf{C}[\mathfrak{S}_d]$ defined by $f_x(a) = xa$. Furthermore we can consider $\mathbf{C}[\mathfrak{S}_d] c_\lambda$ as a right $\mathbf{C}[\mathfrak{S}_d]$-module by defining the action of basis elements $e_g \in \mathbf{C}[\mathfrak{S}_d]$ as

$$a \cdot e_g = e_{g^{-1}} a \tag{3.8}$$

for $a \in \mathbf{C}[\mathfrak{S}_d] c_\lambda$ and extending linearly. The advantage of this is that we now have an isomorphism of right $\mathbf{C}[\mathfrak{S}_d]$-modules

$$\varphi : c_\lambda \mathbf{C}[\mathfrak{S}_d] \xrightarrow{\cong} \mathbf{C}[\mathfrak{S}_d] c_\lambda \tag{3.9}$$

with φ defined on basis elements e_h of $\mathbf{C}[\mathfrak{S}_d]$ by $\varphi(c_\lambda e_h) = e_{h^{-1}} c_\lambda$ and extending linearly. Isomorphims (3.7) and (3.9)now tells us that (3.6) is isomorphic to the $\mathrm{GL}_n(\mathbf{C}) - \mathbf{C}[\mathfrak{S}_d]$ bimodule

$$\bigoplus_\lambda \left(\left(V^{\otimes d} \otimes_{\mathbf{C}[\mathfrak{S}_d]} \mathbf{C}[\mathfrak{S}_d] c_\lambda\right) \otimes_{\mathbf{C}} \mathbf{C}[\mathfrak{S}_d] c_\lambda\right) \tag{3.10}$$

where $\mathrm{GL}_n(\mathbf{C})$ acts on $V^{\otimes d}$ while $\mathbf{C}[\mathfrak{S}_d]$ acts on $\mathbf{C}[\mathfrak{S}_d] c_\lambda$ on the right by the action defined in Equation (3.8). Using Statement (1) of Theorem 3.3, we now have

$$\bigoplus_\lambda \left((V^{\otimes d} \otimes_{\mathbf{C}[\mathfrak{S}_d]} \mathbf{C}[\mathfrak{S}_d] c_\lambda) \otimes_{\mathbf{C}} \mathbf{C}[\mathfrak{S}_d] c_\lambda \right) \cong \bigoplus_\lambda \mathbb{S}_\lambda(V) \otimes_{\mathbf{C}} \mathbf{C}[\mathfrak{S}_d] c_\lambda \tag{3.11}$$

$$= \bigoplus_\lambda \mathbb{S}_\lambda(V) \otimes_{\mathbf{C}} V_\lambda \tag{3.12}$$

by definition of $V_\lambda = \mathbf{C}[\mathfrak{S}_d]c_\lambda$. Since $\mathbb{S}_\lambda(V)$ is zero precisely when the number of parts of λ is greater than $n = \dim V$, the direct sum above is over all partitions λ of d of at most n parts. This proves Statement (4) of Theorem 3.3.

□

4. The character of $\mathbb{S}_\lambda(V)$.

Recall for a representation $\rho : \mathrm{GL}_n(\mathbf{C}) \to \mathrm{GL}(W)$ on a $\mathbf{C}$-vector space W, the character of ρ (denoted χ_ρ) is the complex valued function on $\mathrm{GL}_n(\mathbf{C})$ defined by $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$. In this section, we will denote the character of $\mathbb{S}_\lambda(V)$ by $\chi_{\mathbb{S}_\lambda(V)}$. Before moving on we state the main idea of this character calculation. Consider again the isomorphism

$$V^{\otimes d} \cong \bigoplus_\lambda \mathbb{S}_\lambda(V) \otimes_{\mathbf{C}} V_\lambda$$

in Theorem 3.3. Take some $g \in \mathrm{GL}_n(\mathbf{C})$ and $\sigma \in \mathfrak{S}_d$. Since the actions of g and σ on $V^{\otimes d}$ commute, it makes sense to speak of $g\sigma$ as an endomorphism of this space. Now suppose we know how to calculate the trace of $g\sigma$ on $V^{\otimes d}$. If we know the character of V_λ as well, in theory we should be able to get the character of the Schur functor. This is the way we will proceed.

4.1. **Symmetric Polynomials.**

We need to define some symmetric polynomials for use in this section and in the next. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of d of at most n parts. Recall that $n = \dim V$. First we define the *monomial symmetric polynomials* in n variables as

$$m_\lambda(x_1, \ldots, x_n) := \sum x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \tag{4.1}$$

where the sum is taken over all distinct monomials obtained from $x_1^{\lambda_1} \ldots x_n^{\lambda_n}$ by permuting the variables $x_1, \ldots, x_n$. Next we may also define the *Schur polynomial*

$$s_\lambda(x_1, \ldots, x_n) := \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & \ldots & x_n^{\lambda_1+n-1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ x_1^{\lambda_n+n-n} & \ldots\ldots & x_n^{\lambda_n+n-n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & \ldots & x_n^{n-2} \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{vmatrix}}. \tag{4.2}$$

Notice that the denominator in the expression above is the discriminant

$$\Delta(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

One non-obvious fact from the definition of the Schur polynomial is that it is symmetric in the variables $x_1, \ldots, x_n$. It turns out that these polynomials m_λ and s_λ for λ a partition of d with at most n parts are $\mathbf{Z}$-bases for the degree d component of the $\mathbf{N}$-graded ring

$$\mathbf{Z}_{\mathrm{sym}}[x_1, \ldots, x_n],$$

the ring of symmetric polynomials in n-variables. Lastly, we define the k^{th} *power sum polynomial by*

$$p_k(x_1, \ldots, x_n) = x_1^k + \ldots + x_n^k. \tag{4.3}$$

Now we turn to a formula of Frobenius for calculating the character χ_λ of V_λ. Let $C_{\mathbf{i}}$ denote the conjugacy class in $\mathfrak{S}_d$ determined by a sequence

$$\mathbf{i} = (i_1, \dots, i_d) \text{ with } \sum \alpha i_\alpha = d.$$

Here $C_{\mathbf{i}}$ consists of those permutations that have i_1 1 -cycles, i_2 2- cycles, ... and i_d d-cycles. For example the sequence $(0, 1, 2, 0, 0)$ corresponds to the conjugacy class $(12)(345)$ of $\mathfrak{S}_5$. Then Frobenius proved that the character of $g \in C_{\mathbf{i}}$ is given by

$$\prod_{k=1}^{d} p_k(x_1, \dots, x_n)^{i_k} = \sum_\lambda \chi_\lambda(C_{\mathbf{i}}) s_\lambda(x_1, \dots, x_n) \tag{4.4}$$

where the sum on the right is over all partitions λ of d in at most n parts. The left hand side is an element of the degree d component of $\mathbf{Z}_{\mathrm{sym}}[x_1, \dots, x_n]$ which guarantees that the $\chi_\lambda(C_{\mathbf{i}})$ are all integers. The reader is referred to Chapter 4 of [FH91] for a proof of this formula. We are now ready to calculate the character of the Schur functor. Suppose $g \in \mathrm{GL}_n(\mathbf{C})$ has eigenvalues $\mu_1, \dots, \mu_n$ (including multiplicities). Let $\sigma \in \mathfrak{S}_d$ be in some conjugacy class $C_{\mathbf{i}}$. Suppose for the moment the trace of $g\sigma$ on $V^{\otimes d}$, $\mathrm{Tr}_{V^{\otimes d}}(g\sigma)$ is given by

$$\mathrm{Tr}_{V^{\otimes d}}(g\sigma) = \prod_{k=1}^{d} p_k(\mu_1, \dots, \mu_n)^{i_k}. \tag{4.5}$$

On the other hand by the isomorphism of Statement (4) of Theorem 3.3, we get that

$$\mathrm{Tr}_{V^{\otimes d}}(g\sigma) = \mathrm{Tr}_{\bigoplus_\lambda (\mathbb{S}_\lambda(V) \otimes_{\mathbf{C}} V_\lambda)}(g\sigma) = \sum_\lambda \chi_{\mathbb{S}_\lambda(V)}(g) \chi_{V_\lambda}(C_{\mathbf{i}})$$

because if σ is in $C_{\mathbf{i}}$ then so is σ^{-1}. The sum of course is taken over all λ a partition of d of at most n rows. Comparing this with the Frobenius formula 4.4 gives that

$$\sum_\lambda \chi_{V_\lambda}(C_{\mathbf{i}}) \chi_{\mathbb{S}_\lambda(V)}(g) = \sum_\lambda \chi_\lambda(C_{\mathbf{i}}) s_\lambda(\mu_1, \dots, \mu_n). \tag{4.6}$$

It now follows by character orthogonality for finite groups that the character of the Schur functor is given by the surprisingly simple formula

$$\chi_{\mathbb{S}_\lambda(V)}(g) = s_\lambda(\mu_1, \dots, \mu_n). \tag{4.7}$$

In other words the character is simply the Schur polynomial evaluated at the eigenvalues of g. It remains to verify Equation 4.5. This is a straightforward computation which we say a little on, because it is not very illuminating. One first proves it with $g \in \mathrm{GL}_n(\mathbf{C})$ diagonalizable and σ an element of $\mathfrak{S}_d$ that looks like $(12 \dots\dots k)(k+1)\dots(n)$. Then because the diagonalizable matrices are dense in $\mathrm{GL}_n(\mathbf{C})$, we get for all $g \in \mathrm{GL}_n(\mathbf{C})$ and σ of the form above that Equation 4.5 holds. It is readily verified next that the equation holds when σ is of the form

$$\sigma = (12 \dots\dots k)(k+1 \dots\dots k+j)(k+j+1)(k+j+2)\dots(n)$$

from which the case for general σ in any conjugacy class follows immediately. As a corollary of this character calculation, we have

Corollary 4.1. *The dimension of* $\mathbb{S}_\lambda(V)$ *is equal to*

$$\prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Proof. From Equation 4.2 we get that

$$\mathbb{S}_\lambda(1, x, \dots, x^{n-1}) = x^n \prod_{i<j} \frac{x^{\lambda_i - \lambda_j + j - i} - 1}{x^{j-i} - 1}.$$

Hence

$$\begin{aligned}\dim \mathbb{S}_\lambda(V) = s_\lambda(1,1,\ldots,1) &= \lim_{x\to 1} x^n \prod_{i<j} \frac{x^{\lambda_i-\lambda_j+j-i}-1}{x^{j-i}-1} \\ &= \lim_{x\to 1} \prod_{i<j} \frac{(1+x+\ldots+x^{\lambda_i+\lambda_j-1})}{1+x+\ldots+x^{j-i}} \\ &= \prod_{i<j} \frac{\lambda_i-\lambda_j+j-i}{j-i}.\end{aligned}$$

□

5. The Schur Functor as an Irreducible Representation of $\mathfrak{sl}_n$

We have discussed Schur–Weyl duality and as a consequence of Theorem 3.3 obtained the character of the Schur functor. This section is the final part of this project, where we first discuss some background on the complex Lie algebra $\mathfrak{sl}_n$ and proceed to calculate the highest weight of the Schur functor as a representation of $\mathfrak{sl}_n$.

We recall that the usual Cartan subalgebra of $\mathfrak{sl}_n$ is defined to be the subspace $\mathfrak{h}$ of diagonal matrices whose entries sum to zero. Now given a representation $\pi : \mathfrak{sl}_n \to \mathfrak{gl}(W)$, we say that an element $\mu \in \mathfrak{h}^*$ is a weight with weight vector v if for all $H \in \mathfrak{h}$,

$$\pi(H)v = \mu(H)v.$$

The weights of the adjoint representation are special and so we call them roots. A weight vector for the adjoint representation is thus called a root vector. Of course the roots are the linear functionals $L_i - L_j \in \mathfrak{h}^*$ for $i \neq j, 1 \leq i,j \leq n$. Recall the L_i's are defined by $L_i(\text{diag}(a_1,\ldots,a_n)) = a_i$. We call a root $L_i - L_j$ positive if $j > i$ and negative otherwise. With this, we have the following definition.

Definition 5.1. *A weight μ is said to be of highest weight if its corresponding weight vector is annihilated by all the positive root spaces.*

A representation of $\mathfrak{sl}_n$ is then said to be a *highest weight representation* if there exists a highest weight vector v such that the smallest invariant subspace containing v is the entire representation. By a result from highest weight theory, we know that any irreducible representation of $\mathfrak{sl}_n$ is a highest weight representation. The highest weight vector is unique up to scaling. Conversely any highest weight representation is also irreducible. For proofs of these results, we refer the reader to Chapter 7, [Hal03]. We are now ready to be put an ordering on the weights of an irreducible representation that will be consistent with our notion of highest weight above.

Definition 5.2. *Given weights $\mu_1 = a_1L_1 + \ldots a_nL_n$ and $\mu_2 = b_1L_1 + \ldots b_nL_n$ of some irreducible representation (π, W) of $\mathfrak{sl}_n$, we say that μ_1 is higher than μ_2 (denoted $\mu_1 > \mu_2$) if the first i for which $a_i - b_i$ is non-zero (if any) is positive. A weight μ is then said to be of highest weight if for any other weight ν, $\nu \leq \mu$.*

In Definition 5.2 above, *a priori* it does not make sense to speak of the difference of complex numbers being positive. However we know from the representation theory of $\mathfrak{sl}_n$ that in fact a_i and b_i are always *integers* when the representation is irreducible so this does make sense. We note also that this order is a total ordering on the weights. Now we need to see that the definition of highest weight from Definition 5.1 is consistent with the ordering from Definition 5.2. To see this suppose $\pi : \mathfrak{sl}_n \to \mathfrak{gl}(W)$ is an irreducible representation of $\mathfrak{sl}_n$. Suppose that $\mu = a_1L_1 + \ldots a_nL_n$ is a weight of a weight vector v that is not eliminated by all the positive root vectors. Then there is a root $L_i - L_j$ for some i,j with $j > i$ for which we can choose X a root vector corresponding to $L_i - L_j$ such that $\pi(X)v \neq 0$. But now we find that $\pi(H)\pi(X)(v) = (a_1L_1 + \ldots a_nL_n + L_i - L_j)(H)\pi(X)v$ for all $H \in \mathfrak{h}$. With respect to the ordering defined in Definition 5.2, we find

$$a_1L_1 + \ldots a_nL_n + L_i - L_j > a_1L_1 + \ldots a_nL_n$$

and so μ cannot be of highest weight. The converse follows similarly upon noting that if v is a weight vector eliminated by all the positive root spaces, then any other $w \in W$ is a linear combination of elements the form

$$\pi(Y_1)\pi(Y_2)\ldots\pi(Y_N)v$$

for $Y_1, \ldots, Y_N$ some negative root vectors.

Although we have proven the equivalence of the two definitions only in the case of an irreducible representation, this is all we need. Now suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of some positive integer d. In the case when $W = V = \mathbf{C}^n$, we have seen that the Schur functor $\mathbb{S}_\lambda(V)$ is irreducible as a $\mathrm{GL}_n(\mathbf{C})$-representation. Hence it is irreducible as an $\mathrm{SL}_n(\mathbf{C})$-representation because any element in $\mathrm{GL}_n(\mathbf{C})$ is a scalar multiple of an element in $\mathrm{SL}_n(\mathbf{C})$. Since the latter is a connected Lie group (in fact simply connected), the Schur functor defines an irreducible representation of $\mathfrak{sl}_n$.

Proposition 5.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of d in exactly $n = \dim V$ parts. The representation $\mathbb{S}_\lambda(V)$ is the irreducible representation of $\mathfrak{sl}_n$ up to isomorphism with highest weight $\lambda_1 L_1 + \ldots + \lambda_n L_n$.*

Proof. For the moment let Φ be any representation of $\mathrm{SL}_n(\mathbf{C})$ on $\mathrm{GL}(W)$ for some complex vector space W of finite dimension. Let ϕ be the induced representation of $\mathfrak{sl}_n$ on $\mathfrak{gl}(W)$. Now by $\mathfrak{sl}_n$ theory we may decompose W as a direct sum of weight spaces $\bigoplus_{\alpha \in \mathfrak{h}^*} W_\alpha$. Choose some weight vector $w_\alpha \in W_\alpha$. Then by commutativity of

$$\begin{array}{ccc} \mathrm{SL}_n(\mathbf{C}) & \xrightarrow{\Phi} & \mathrm{GL}(W) \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{sl}_n & \xrightarrow{\phi} & \mathfrak{gl}(W) \end{array}$$

we get that for any $H \in \mathfrak{h}$, $\Phi(\exp(H)) = \exp(\phi(H))$. Furthermore given any diagonal matrix $A \in \mathrm{SL}_n(\mathbf{C})$ it is clear we can write $A = \mathrm{diag}(e^{x_1}, \ldots, e^{x_n})$ for some $x_1, \ldots, x_n \in \mathbf{C}$. Write $H = \mathrm{diag}(x_1, \ldots, x_n)$ so that $A = \exp(H)$. Expanding the exponential as a power series gives

$$\Phi(A) w_\alpha = e^{\alpha(H)} w_\alpha.$$

If we identify a weight $\alpha = \alpha_1 L_1 + \ldots + \alpha_n L_n \in \mathfrak{h}^*$ with the n-tuple $(\alpha_1, \ldots, \alpha_n)$ of integers, we have

$$\mathrm{Tr}(\Phi(A)) = \sum_{\alpha \in \mathfrak{h}^*} (\dim W_\alpha) e^{\alpha(H)} \tag{5.1}$$

$$= \sum_{(\alpha_1, \ldots, \alpha_n) \in \mathfrak{h}^*} (\dim W_{(\alpha_1, \ldots, \alpha_n)}) e^{(\alpha_1 L_1 + \ldots + \alpha_n L_n)(H)} \tag{5.2}$$

$$= \sum_{(\alpha_1, \ldots, \alpha_n) \in \mathfrak{h}^*} (\dim W_{(\alpha_1, \ldots, \alpha_n)}) (e^{x_1})^{\alpha_1} \ldots (e^{x_n})^{\alpha_n}. \tag{5.3}$$

Suppose now that Φ is the representation of $\mathrm{SL}_n(\mathbf{C})$ on $W = \mathbb{S}_\lambda(V)$. Then from the Equation 4.7 we get that $\mathrm{Tr}(\Phi(A)) = s_\lambda(e^{x_1} \ldots, e^{x_n})$. By [FH91, Equation (A.19)] we can write

$$s_\lambda(e^{x_1}, \ldots, e^{x_n}) = m_\lambda(e^{x_1}, \ldots, e^{x_n}) + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu(e^{x_1}, \ldots, e^{x_n})$$

where the λ, μ are partitions of d and m_λ, m_μ as defined in Equation 4.1. The sum is taken over all partitions $\mu = (\mu_1, \ldots, \mu_n)$ such that each μ is less than λ with respect to the following ordering: We say that $\lambda = (\lambda_1, \ldots \lambda_n)$ is greater than $(\mu_1, \ldots, \mu_n)$ if the first non-vanishing $\lambda_i - \mu_i$ is positive. The integers $K_{\lambda\mu}$ are called Kostka numbers and are defined combinatorially as the number of ways to fill the boxes of the diagram for λ with μ_1 $1's$, μ_2 $2's$, $\ldots \mu_n$ $n's$. In particular,

$$K_{\lambda\lambda} = 1 \quad \text{and} \quad K_{\lambda\mu} = 0 \text{ for } \mu > \lambda.$$

In addition $K_{\lambda\mu} = 0$ if μ has more non-zero terms than λ. The above discussion shows we have found two ways of calculating the trace of $\Phi(A)$ and hence they must be equal. In other words

$$\sum_{(\alpha_1, \ldots, \alpha_n) \in \mathfrak{h}^*} \dim W_{(\alpha_1, \ldots, \alpha_n)} (e^{x_1})^{\alpha_1} \ldots (e^{x_n})^{\alpha_n} = m_\lambda(e^{x_1}, \ldots, e^{x_n}) + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu(e^{x_1}, \ldots, e^{x_n})$$

must hold for all $x_1, \ldots, x_n$ that sum to zero. We are thus forced to conclude that any weight

$$\alpha_1 L_1 + \ldots \alpha_n L_n = \mu_{i_1} L_1 + \ldots + \mu_{i_n} L_n$$

for $i_1, \ldots, i_n$ distinct positive integers from 1 to n. The key point now is that our ordering on partitions is *mutadis mutandis* the same as the ordering on weights give in Definition 5.2. Hence the highest of the weights that appears with respect to Definition 5.2 is $\lambda_1 L_1 + \ldots + \lambda_n L_n$ and the proposition is proven. $\square$

Corollary 5.4. *The irreducible representation of $\mathfrak{sl}_n$ of highest weight*

$$a_1 L_1 + \ldots a_{n-1}(L_1 + \ldots + L_{n-1})$$

with a_i all positive integers is $\mathbb{S}_\lambda(V)$ for $\lambda = (a_1 + \ldots + a_{n-1}, a_2 + \ldots + a_{n-1}, \ldots, a_{n-1}, 0)$.

Corollary 5.5. *The dimension of the irreducible representation of highest weight above is*

$$\prod_{1 \le i < j \le n} \frac{(a_i + \ldots + a_{j-i}) + j - i}{j - i}.$$

As a final note, because $\mathrm{SL}_n(\mathbf{C})$ is simply connected, there is a bijection between the irreducible representations of $\mathrm{SL}_n(\mathbf{C})$ and that of its Lie algebra. Thus we have also proven that any irreducible representation of $\mathrm{SL}_n(\mathbf{C})$ is isomorphic to $\mathbb{S}_\lambda(V)$, for some partition $\lambda = (a_1 + \ldots + a_{n-1}, a_2 + \ldots + a_{n-1}, \ldots, a_{n-1}, 0)$ corresponding to a choice of positive integers $a_1, \ldots, a_{n-1}$.

References

[Bum04] Daniel Bump, *Lie groups*, Springer-Verlag, New York,USA, 2004.

[FH91] William Fulton and Joe Harris, *Representation theory: A first course*, Springer - Verlag, New York, USA, 1991.

[Ful97] William Fulton, *Young tableaux: with applications to representation theory and geometry*, Cambridge University Press, Cambridge, United Kingdom, 1997.

[Hal03] Brian Hall, *Lie groups, lie algebras and representations: An elementary introduction*, Springer-Verlag, New York, USA, 2003.

[Pro07] Claudio Procesi, *Lie groups: An approach through invariants and representations*, Springer Science, New York, USA, 2007.

[Ser97] Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag, New York,USA, 1997.