A SEMICONTINUITY THEOREM FOR WEIGHTS - SÉMINAIRE DELIGNE-LAUMON

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1. Weights

We fix the following notation. We will denote by $k = \mathbf{F}_q$ the finite field with q elements, $\overline{k}$ an algebraic closure of k and k_n a degree n extension of k (necessarily isomorphic to $\mathbf{F}_{q^n}$). In this article, we will refer to a finite type scheme X_0/k as simply a scheme. Furthermore, we will refer to a Weil sheaf $\mathscr{G}_0$ on X_0 as simply a sheaf. By convention, such a sheaf is always constructible, viz isomorphic in the Artin-Rees category to a strictly constructible sheaf. We will fix an isomorphism $\tau : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$. For a closed point $x \in |X_0|$, we will denote by $d(x)$ the degree of the field extension $k(x)/k$, and $N(x)$ the cardinality of $k(x)$. Recall the following definition of purity:

Definition 1.0.1. Let β be a real number.

(1) Choose a $\overline{k}$-point $\overline{x} \in X$ lying over $x \in |X_0|$. The Weil group $W(\overline{k}/k(x))$ acts on the stalk at $\mathscr{G}_{0\overline{x}}$ via the geometric Frobenius $F_x : \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$. We say that $\mathscr{G}_0$ is **τ-pure** of weight β if for every $x \in |X_0|$, and all eigenvalues $\alpha \in \overline{\mathbf{Q}}_\ell$ of F_x, we have
$$|\tau(\alpha)| = N(x)^{\beta/2}.$$

If $\mathscr{G}_0$ is a general Weil sheaf, not necessarily pure, we would still like to have the notion of the weight of $\mathscr{G}_0$. This brings us to the following definition.

Definition 1.0.2. For a scheme X_0/k and sheaf $\mathscr{G}_0$ on X_0, we define the **maximal weight** of $\mathscr{G}_0$ (with respect to τ) as
$$w(\mathscr{G}_0) := \sup_{x \in |X_0|} \sup_{\alpha \text{ eigenvalue}} \frac{\log(|\tau(\alpha)|^2)}{\log N(x)}.$$
For reasons of convention, we define the weight of the zero sheaf to be $-\infty$.

2. Convergence of the L-function

We show in this section that the weight of a Weil sheaf controls the convergence of its L-function.

Lemma 2.0.1. Let X_0/k be a scheme. Then we have the estimate
$$|X_0(k_n)| = O(q^{n \dim X_0})$$
as $n \to \infty$.

Proof. We have $|X_0(k_n)| = |X_{0\mathrm{red}}(k_n)|$ and so we can reduce to the case that X_0 is reduced. By the principle of inclusion-exclusion, we can reduce to the case where X_0 is integral. Then by Noether normalization, there is an open dense subset $U_0 \subseteq X_0$ with a finite morphism $f : U_0 \to \mathbf{A}_{k_n}^{\dim X_0}$. Hence we obtain
$$|U(k_n)| \leq (\deg f)(\# k_n)^{\dim X_0} = (\deg f) q^{n \dim X_0}.$$
The result follows by induction on dimension, since $\dim(X_0 \setminus U_0) < \dim X_0$. □

Lemma 2.0.2. Let V be a finite dimensional vector space and F an endomorphism of V, and $d \in \mathbf{N}$ a non-negative integer. Then
$$\frac{d}{dt} \log \det(1 - t^d F|V)^{-1} = \sum_{n \geq 1} \mathrm{Tr}(F^n) d t^{dn-1}.$$

Proof. Recall that Niccolò introduced the formula

$$\det(1 - t^d F|V)^{-1} = \exp\left(\sum_{n\geq 1} \operatorname{Tr}(F^n)\frac{t^{dn}}{n}\right)$$

in his talk. Taking derivatives, we get

$$\begin{aligned}\frac{d}{dt}\det(1 - t^d F|V)^{-1} &= \left(\sum_{n\geq 1} \operatorname{Tr}(F^n)(dn)\frac{t^{dn-1}}{n}\right)\cdot \exp\left(\sum_{n\geq 1} \operatorname{Tr}(F^n)\frac{t^{dn}}{n}\right) \\ &= \left(\sum_{n\geq 1} \operatorname{Tr}(F^n) d t^{dn-1}\right)\det(1 - t^d F|V)^{-1}\end{aligned}$$

and hence the result. ☐

Proposition 2.0.1. Let $\mathscr{G}_0$ be a sheaf on X_0 and β a real number such that $w(\mathscr{G}_0) \leq \beta$. Then the L-function

$$\tau L(X_0, \mathscr{G}_0, t) = \prod_{x\in|X_0|} \tau \det(1 - t^{d(x)} F_x, \mathscr{G}_{0\overline{x}})^{-1}$$

converges for all $|t| < q^{-\beta/2-\dim X_0}$ and has no zeroes or poles in this region.

Proof. The idea is that we can detect the poles an zeroes of the L-function by looking at its logarithmic derivative. This is because the logarithmic derivative of a complex valued function has poles precisely where the original function has poles or zeroes. We will suppress the isomorphism $\tau : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ in the following for brevity.

$$\begin{aligned}\frac{d}{dt}\log L(X_0, \mathscr{G}_0, t) &= \sum_{x\in|X_0|}\frac{d}{dt}\log\left(\det(1 - t^{d(x)}F_x|\mathscr{G}_{0\overline{x}})^{-1}\right) \\ &= \sum_{x\in|X_0|}\sum_{n\geq 1} d(x)(\operatorname{Tr}(F_x^n))t^{d(x)n-1} \\ &= \sum_{n\geq 1}\left(\sum_{x\in|X_0|:d(x)|n} d(x)(\operatorname{Tr}(F_x^{n/d(x)}))\right)t^{n-1}.\end{aligned}$$

We passed from the first to second line using Lemma 2.0.2. By assumption on the bound of the Frobenius eigenvalues, we have

$$|\operatorname{Tr}(F_x^{n/d(x)})| \leq r q^{n\beta/2}$$

where

$$r := \max_{x\in|X_0|}\dim_{\overline{\mathbf{Q}}_\ell}\mathscr{G}_{0\overline{x}}.$$

Hence

$$\begin{aligned}\frac{d}{dt}\log L(X_0, \mathscr{G}_0, t) &= \sum_{n\geq 1}\left(\sum_{x\in|X_0|:d(x)|n} d(x)(\operatorname{Tr}(F_x^{n/d(x)}))\right)t^{n-1} \\ &\leq \sum_{n\geq 1}\left(\sum_{x\in|X_0|:d(x)|n} d(x)\cdot(r q^{n\beta/2})\right)t^{n-1} \\ &= \sum_{n\geq 1}|X_0(k_n)|\cdot(r q^{n\beta/2})t^{n-1}.\end{aligned}$$

By Lemma 2.0.1, we see that the logarithmic derivative converges for all $|t| < q^{-\beta/2-\dim X_0}$. Therefore $L(X_0, \mathscr{G}_0, t)$ also converges for $|t| < q^{-\beta/2-\dim X_0}$. ☐

3. Semicontinuity of Weights

The motivating question is the following. Suppose $\mathscr{G}_0$ is a sheaf on a scheme X_0/k. Given an open dense $j_0 : U_0 \to X_0$, how does the weight of $\mathscr{G}_0$ compare to that of $j_0^*\mathscr{G}_0$? It turns out that under certain hypotheses on $\mathscr{G}_0$ (made precise below), this is always true. This result will be used in future arguments involving Noetherian induction on X_0. First, we consider the case of curves.

Given a Weil sheaf $\mathscr{G}_0$ on a smooth curve X_0/k, we recall the following facts concerning $H_c^0(X, \mathscr{G})$ and $H_c^2(X, \mathscr{G})$.

(1) If $\mathscr{G}_0$ is lisse, corresponding to some representation V of $\pi_1(X, x)$, then

$$H^0(X, \mathscr{G}) = V^{\pi_1(X,x)}.$$

(2) If X_0 is geometrically irreducible and $U_0 \subseteq X_0$ is an open dense subset, we have

$$H_c^2(X, \mathscr{G}) = H_c^2(U, \mathscr{G}).$$

Indeed, consider the excision sequence

$$0 \to j_! j^*\mathscr{G} \to \mathscr{G} \to i_* i^* \mathscr{G} \to 0$$

associated to the inclusion of spaces

$$\begin{array}{ccc} & & X \setminus U \\ & & \big\downarrow{\scriptstyle i} \\ U & \xrightarrow{\ j\ } & X \end{array}$$

Choose a compactification $j' : X \to \overline{X}$ and apply $j'_!$ to the exact sequence above to get

$$0 \to j'_! j_! j^* \mathscr{G} \to j'_! \mathscr{G} \to j'_! i_* i^* \mathscr{G} \to 0.$$

Since the sheaf $j'_! i_* i^* \mathscr{G}$ is supported on the finite set of closed points $X \setminus U$, it is enough to prove that

$$H^i(\overline{X}, j'_! i_* i^* \mathscr{G}) = 0$$

for $i = 1, 2$. This follows from the fact that the higher étale cohomology of a separably closed field is zero.

Lemma 3.0.1. Let X_0/k be a geometrically irreducible affine curve, $j_0 : U_0 \to X_0$ an open immersion of an open subscheme and $\mathscr{G}_0$ a sheaf on X_0 such that the canonical adjunction map $\mathscr{G}_0 \to j_{0*} j_0^* \mathscr{G}_0$ is an isomorphism. Assume further that $j_0^* \mathscr{G}_0$ is lisse. Then

$$H_c^0(X, \mathscr{G}) = 0.$$

Proof. Let $Z \subseteq X$ be a complete subvariety and define $V := X \setminus Z$. Then $V \neq \emptyset$ because X is not complete. We have to show that

$$H_Z^0(X, \mathscr{G}) := \ker\left(H^0(X, \mathscr{G}) \to H^0(V, \mathscr{G}|_V)\right)$$

is zero. Since $\mathscr{G}_0 \to j_{0*} j_0^* \mathscr{G}_0$ is an isomorphism, we may rewrite this as

$$H_Z^0(X, \mathscr{G}) = \ker\left(H^0(U, \mathscr{G}|_U) \to H^0(V \cap U, \mathscr{G}|_{U \cap V})\right).$$

The intersection $U \cap V$ is non-empty for X is irreducible. Let η denote the generic point of U. Since $\mathscr{G}|_U$ is lisse, for any $u \in U$ the specialization map $\mathscr{G}_{0\overline{u}} \to \mathscr{G}_{0\overline{\eta}}$ is an isomorphism. This implies that any section that vanishes on $V \cap U$ also vanishes on U, consequently $H_Z^0(X, \mathscr{G}) = 0$ as desired. ☐

Proposition 3.0.1 (Semicontinuity of Weights for Curves)**.** Let X_0/k be a geometrically irreducible smooth curve. Let $j_0 : U_0 \hookrightarrow X_0$ be an affine open and $\mathscr{G}_0$ a lisse sheaf on U_0. We define $S_0 := X_0 \setminus U_0$. Suppose $\mathscr{G}_0$ is a sheaf on X_0 that satisfies the following conditions:

(1) $j_0^* \mathscr{G}_0$ is lisse.
(2) $H_S^0(X, \mathscr{G}) = 0$.

Then

$$w(j_0^* \mathscr{G}_0) \leq \beta \implies w(\mathscr{G}_0) \leq \beta.$$

The idea here is the following. We first reduce to the case where $H^0_c(X, \mathscr{G}) = 0$ by the previous lemma. Therefore by the Grothendieck-Lefschetz trace formula, the only contribution to the poles of the L-function will come from $H^2_c(X, \mathscr{G})$. This allows us to show that $L(X_0, \mathscr{G}_0, t)$ has no poles outside of the disk $|t| < q^{-\beta/2-1}$. On the other hand, by assumption the L-function on U_0 converges and has no zeroes in the same region. So writing

$$L(X_0) = L(U_0)L(S_0),$$

and noticing that $L(S_0)$ has only finitely many factors, we deduce immediately bounds on the frobenius eigenvalues of $\mathscr{G}_{0\overline{s}}$ for every $s \in |S_0|$. The result follows by a trick of Deligne of considering higher tensor powers of $\mathscr{G}_0$.

Proof. By removing a point from U_0, we may assume that X_0 is affine. The assumption $H^0_S(X, \mathscr{G}) = 0$ implies $\mathscr{G}_0 \hookrightarrow j_{0*}j_0^*\mathscr{G}_0$. In this case

$$w(\mathscr{G}_0) \leq w(j_{0*}j_0^*\mathscr{G}_0)$$

and so we can reduce to the case where $\mathscr{G}_0 \to j_{0*}j_0^*\mathscr{G}_0$ is an isomorphism. Then by Lemma 3.0.1 and the Grothendieck-Lefschetz trace formula, we have

$$L(X_0, \mathscr{G}_0, t) = \frac{\det(1 - Ft|H^1_c(X, \mathscr{G}))}{\det(1 - Ft|H^2_c(X, \mathscr{G}))}.$$

Define $\mathscr{F}_0 := j_0^*\mathscr{G}_0$. For $u \in |U_0|$, this corresponds to a representation V of $\pi_1(U, \overline{u})$. Then

$$\begin{aligned}
H^2_c(X, \mathscr{G}) &= H^2_c(U, \mathscr{F}) \\
&= H^0(U, \check{\mathscr{F}}(1))^\vee && \text{(Poincaré duality)} \\
&= H^0(U, \check{\mathscr{F}} \otimes \overline{\mathbf{Q}}_\ell(1))^\vee \\
&= H^0(U, \check{\mathscr{F}})^\vee \otimes \overline{\mathbf{Q}}_\ell(-1) && \text{(Künneth formula)} \\
&= (V^{\pi_1(U,\overline{u})})^\vee(-1) \\
&= (V_{\pi_1(U,\overline{u})})(-1).
\end{aligned}$$

It follows that the poles of $L(X_0, \mathscr{G}_0, t)$ are of the form $1/\alpha q$ where α is an eigenvalue of F_u on $V_{\pi_1(U,\overline{u})}$ (recall geometric Frobenius acts by q^{-1} on $\overline{\mathbf{Q}}_\ell(1)$). Now from the definition of coinvariance, $\alpha^{d(u)}$ lifts to an eigenvalue on V. Therefore by the assumption $w(\mathscr{F}_0) \leq \beta$, we have that $|\tau(\alpha^{d(u)})| \leq q^{d(u)\beta/2}$, i.e.

$$\left|\tau\left(\frac{1}{\alpha q}\right)\right| > q^{-\beta/2-1}$$

and so $L(X_0, \mathscr{G}_0, t)$ converges for $|t| < q^{-\beta/2-1}$.

On the other hand, we may write

$$L(X_0, \mathscr{G}_0, t) = L(U_0, j_0^*\mathscr{G}_0, t) \prod_{s_0 \in |S_0|} \det(1 - F_s t^{d(s)}, \mathscr{G}_{0\overline{s}})^{-1}.$$

The asumption $w(j_0^*\mathscr{G}) \leq \beta$ implies that the factor $L(U_0, j_0^*\mathscr{G}, t)$ converges and has no zeroes for $|t| < q^{-\beta/2-1}$. Therefore since $|S_0|$ is finite it follows that none of the factors

$$\det(1 - F_s t^{d(s)}, \mathscr{G}_{0\overline{s}})^{-1}$$

has poles for $|t| < q^{-\beta/2-1}$, which implies the estimate

$$|\tau(\tilde{\alpha})| \leq q^{-\beta/2-1}$$

for $\tilde{\alpha}$ an eigenvalue of $F_s : \mathscr{G}_{0\overline{s}} \to \mathscr{G}_{0,\overline{s}}$. Finally, by considering the sheaves $j_{0*}\mathscr{F}^{\otimes k}$ we get the estimate $|\tau(\tilde{\alpha})| \leq q^{-\beta/2-1/k}$. Since this is true for every k, we are done. □

Corollary 3.0.1 (Semicontinuity of Weights for general X_0)**.** Let $\mathscr{G}_0$ be a lisse sheaf on a geometrically irreducible scheme X_0/k and let $j_0 : U_0 \to X_0$ be the inclusion of an open dense subscheme of X_0. Then

$$w(\mathscr{G}_0) = w(j_0^*\mathscr{G}_0).$$

Proof. By taking the normalization of $X_{0\mathrm{red}}$ we can reduce to the case where X_0 is a normal geometrically integral scheme. If $\dim X_0 = 1$ we are done by the semicontinuity theorem for curves above. If $\dim X_0 > 1$, we may connect any point of $X_0 \setminus U_0$ to a point of U_0 by a curve, and conclude again using the semicontinuity theorem for curves above. The assumption that $\mathscr{G}_0$ is lisse on X_0 is used to say that $H^0_{X\setminus U}(X, \mathscr{G}) = 0$ in order to apply the previous lemma. □

References

[1] Deligne, Pierre. La Conjecture de Weil: II. *Inst. Hautes Études Sci. Publ. Math.* No. 52 (1980), 137-252.
[2] Kiehl, Reinhardt and Weissauer, Rainer. Weil conjectures, perverse sheaves and l'adic Fourier transform. *Springer-Verlag, Berlin,* 2001. xii+375 pp. ISBN: 3-540-41457-6