A PROPER SCHEME WITH INFINITE-DIMENSIONAL FPPF COHOMOLOGY

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1. Introduction

In algebraic geometry, very often one encounters theorems of the following flavor:

Theorem 1.1. *Let $f : X \to S$ be a proper morphism of spaces. Then for every sheaf $\mathcal{F}$ on X that is finite, so is its pushforward $Rf_*\mathcal{F}$.*

Notice how I was being deliberately vague in the theorem above. What are X and Y? What does "finite" mean? Well, it turns out that this depends on the context. In the setting of coherent cohomology, "finite" should mean coherent. On the other hand, in étale cohomology "finite" should mean constructible. Now unfortunately (or fortunately?) I am not a number theorist, so to me I'll take constructible to mean "a finite set." So in this case, finite really means - as you guessed it - finite. Let me now give two theorems (in the coherent and étale setting) that illustrate this:

Theorem 1.2. *Let $f : X \to S$ be a proper morphism of locally Noetherian schemes. Let $D^+_{\text{coh}}(X)$ denote the derived category of bounded above $\mathcal{O}_X$-modules with coherent cohomology. Then for any $\mathcal{F} \in D^+_{\text{coh}}(X)$, the direct image $Rf_*\mathcal{F} \in D^+_{\text{coh}}(S)$.*

Theorem 1.3. *Let $f : X \to S$ be a proper morphism of locally Noetherian schemes. Let $D^+_{\text{c}}(X)$ denote the derived category of bounded above abelian sheaves (in the étale topology) with constructible cohomology. Then for any $\mathcal{F} \in D^+_{\text{c}}(X)$, the direct image $Rf_*\mathcal{F} \in D^+_{\text{c}}(S)$.*

Remark 1.4. I do not know/remember if the Noetherian assumption is necessary in Theorem 1.3, but it certainly is for Theorem 1.2, because the proof of Theorem 1.2 is by dévissage on the category of coherent sheaves on X.

In this article, I will show that there is no such analogue of a finiteness theorem in the fppf topology:

Theorem 1.5. *Let k be an algebraically closed field of characteristic p. Then*

$$H^2_{\text{fppf}}(X, \mu_p) \simeq k^+ \times \mathbf{Z}/p\mathbf{Z}.$$

Remark 1.6. In the étale topology, note that $H^2(X, \mu_p) = 0$. The reason is because μ_p on $X_{\text{ét}}$, for X a reduced, $\mathbf{F}_p$-algebra the trivial sheaf. Indeed, let $U \to X$ be any étale open. Then since X is reduced, so is U. Now any $f \in \mu_p(U)$ satisfies $f^p = 1$. But in characteristic p, this says $(f-1)^p = 0$, hence $f = 1$ since U is reduced.

2. The Brauer group of a singular curve

We first prove a preliminary result which concerning the Brauer group of a singular curve. All cohomology considered will be in the étale topology.

Lemma 2.1. *Let k be an algebraically closed field (of any characteristic) and X/k a reduced, possibly singular curve. Then $H^2(X, \mathbf{G}_m) = 0$.*

Proof. First we claim we may assume that X is reduced. Indeed, for a closed subscheme $j : Y \subseteq X$ defined by an ideal $\mathcal{I}$ with $\mathcal{I}^2 = 0$, we also have $H^2(X, \mathbf{G}_m) \hookrightarrow H^2(Y, \mathbf{G}_m)$. Indeed, consider the exact sequence of étale sheaves

$$0 \to 1 + \mathcal{I} \to \mathbf{G}_m \to j_*\mathbf{G}_m \to 0.$$

Taking cohomology, we see that a sufficient condition for $H^2(X, \mathbf{G}_m) \to H^2(Y, \mathbf{G}_m)$ to be injective is that $H^2(X, 1+\mathcal{I}) = 0$. Observe that $1+\mathcal{I} \simeq \mathcal{I}$ via the map $i \mapsto 1+i$. But now $\mathcal{I}$ is a coherent sheaf, and therefore the cohomology $H^2(X, \mathcal{I})$ may be taken to be coherent. Since X is a curve, it follows that $H^2(X, \mathcal{I}) = 0$.

The claim now follows from the fact that the nilradical of X admits a filtration with successive quotients square-zero ideals in $\mathcal{O}_X$.

Let $\pi : \widetilde{X} \to X$ be the normalization morphism. We have a Leray spectral sequence with $E_2^{p,q}$ term $H^p(X, R^q\pi_*\mathbf{G}_m)$ converging to $E_\infty^{p+q} := H^{p+q}(\widetilde{X}, \mathbf{G}_m)$. Consider the second abutment $E_\infty^2 = H^2(\widetilde{X}, \mathbf{G}_m)$. Then $\widetilde{X}$, being a (possibly union) of smooth curve(s) over an algebraically closed field k, must have

$$E_\infty^2 = H^2(\widetilde{X}, \mathbf{G}_m) = 0$$

by Tsen's theorem.

Now I claim there is an injection $E_2^{2,0} \hookrightarrow E_\infty^2$, and consequently that $E_2^{2,0} = H^2(X, \pi_*\mathbf{G}_m)$ is zero. Indeed, since π is finite, $R^1\pi_*\mathbf{G}_m$ is zero, and therefore $E_2^{0,1} = 0$, i.e. $E_3^{2,0} = E_2^{2,0}/\mathrm{Im}(E_2^{0,1} \to E_2^{2,0}) = E_2^{2,0}$. It is now easy to see that in fact

$$E_2^{2,0} = E_3^{2,0} = \ldots = E_\infty^{2,0}$$

and hence $E_2^{2,0} \hookrightarrow E_\infty^2$. Finally, I claim that

$$H^2(X, \mathbf{G}_m) = H^2(\widetilde{X}, \pi_*\mathbf{G}_m)$$

which is sufficient to prove the lemma. Indeed, the exact sequence

$$0 \to \mathbf{G}_m \to \pi_*\mathbf{G}_m \to Q \to 0$$

has cokernel Q supported on the singular locus of X. By [Stacks, Tag 056V], the singular locus is a proper closed subset in X, and therefore consists of a finite union of k-valued points. But the higher étale cohomology of an algebraically closed (even separably closed!) field is zero, so $H^1(X, Q) = H^2(X, Q) = 0$. This proves that $H^2(X, \mathbf{G}_m) = H^2(\widetilde{X}, \pi_*\mathbf{G}_m)$ and we win. □

3. Proof of Theorem 1.5

From the Kummer sequence, which is exact in the fppf topology, we obtain an exact sequence

$$0 \to \frac{H^1_{\mathrm{fppf}}(X, \mathbf{G}_m)}{p \cdot H^1_{\mathrm{fppf}}(X, \mathbf{G}_m)} \to H^2_{\mathrm{fppf}}(X, \mu_p) \to H^2_{\mathrm{fppf}}(X, \mathbf{G}_m)[p] \to 0.$$

But $\mathbf{G}_m$ is smooth, and therefore by [BrIII, Theorem 11.7], the cohomology of $\mathbf{G}_m$ in either the étale or fppf topology is the same. It follows by Lemma 1 that

$$H^2_{\mathrm{fppf}}(X, \mu_p) \simeq \frac{H^1_{\mathrm{fppf}}(X, \mathbf{G}_m)}{p \cdot \ H^1_{\mathrm{fppf}}(X, \mathbf{G}_m)} \simeq \frac{\mathrm{Pic}(X)}{p \cdot \mathrm{Pic}(X)}.$$

Theorem 1.5 now follows from the following result, noting that k^+ is *not* p-divisible in characteristic p.

Proposition 3.1. *Let k be an algebraically closed field of characteristic p, and let X/k be the cuspidal cubic. There is an exact sequence of abelian groups*

$$0 \to k^+ \to \mathrm{Pic}(X) \to \mathbf{Z} \to 0.$$

Proof. As in Lemma 2.1, let $\pi : \widetilde{X} \to X$ denote the normalization morphism. Consider the long exact sequence in cohomology obtained from

$$0 \to \mathbf{G}_m \to \pi_*\mathbf{G}_m \to Q \to 0. \quad (1)$$

In other words,

$$1 \to k^\times \to k^\times \to H^0(X, Q) \to \mathrm{Pic}(X) \to H^1(X, \pi_*\mathbf{G}_m) \to 0.$$

By a similar argument using the Leray spectral sequence in Lemma 2.1, and using the fact that $\widetilde{X} \simeq \mathbf{P}^1$, we get

$$H^1(X, \pi_*\mathbf{G}_m) = \mathrm{Pic}(\widetilde{X}) = \mathbf{Z}.$$

Therefore, it remains to show $H^0(X, Q) = k^+$. Why? Because the map $k^\times \to k^\times$ is simply the map on constants, and hence is the identity!

Since Q is supported at the cusp, to compute $H^0(X, Q)$ it is enough to describe the stalk of Q at the cusp $x_0 \in X$. We must now work in coordinates: Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k\{t^2, t^3\}^\times & \longrightarrow & k\{t\}^\times & \longrightarrow & Q_{x_0} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & k[\![t^2, t^3]\!]^\times & \longrightarrow & k[\![t]\!]^\times & \longrightarrow & k^+ & \longrightarrow & 0 \end{array}$$

where $k\{t^2, t^3\}$ is the strict henselization of $k[t^2, t^3]$ at the origin. The top row of this diagram is the stalk of (1) at $x_0 \in X$ (in the étale topology). The maps α, β are the ones obtained from the universal property of the henselization (note the henselization and strict henselization are the same since the residue field is algebraically closed). The map γ is the induced map on quotients, which exists since since all α, β (on the level of rings) are local homomorphisms. the projection map $k[\![t]\!] \to k^+$ is given by

$$f(t) \mapsto \frac{d}{dt} \log f(t)\big|_{t=0}.$$

Note the logarithmic derivative is well-defined because $f(t) \in k[\![t]\!]$ implies that $f(0) \neq 0$.

It is sufficient to show that the map γ is an isomorphism. The key thing we must show is that γ is surjective (injectivity is kinda clear). Now any $a_1 \in k^+$ is hit by the polynomial $f(t) := 1 + a_1 t \in k[\![t]\!]^\times$. But this polynomial lies in $k\{t\}^\times$, i.e. is algebraic. Why? It satisfies $p(t, f(t)) = 0$, where

$$p(x, y) = 1 + a_1 x - y.$$

Hence γ is surjective and we win. □

References

[BrIII] Alexander Grothendieck, *Le groupe de Brauer. III. Exemples et compléments*, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 88–188. MR 244271

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