THE PICARD NUMBER OF A KUMMER SURFACE

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1. Introduction

Let k be a separably closed field of characteristic not 2, and A/k an abelian surface. Then it is a basic fact (e.g. see [Huy16, Example 1.3 (iii)]) that one can make a K3 surface out of A. The construction is as follows. Consider the involution $\iota : A \to A$ given by $x \mapsto -x$. The fixed locus of this involution is exactly $A[2]$, a finite constant closed k-subgroup scheme of A with 16 k-points. Let $Z := A[2]$, and consider the blow-up $p : \mathrm{Bl}_Z A \to A$. By the universal property of the blow-up, the involution ι lifts to a *map* $\tilde{\iota} : \mathrm{Bl}_Z A \to \mathrm{Bl}_Z A$ such that the diagram

$$\begin{array}{ccc} \mathrm{Bl}_Z A & \xrightarrow{\tilde{\iota}} & \mathrm{Bl}_Z A \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{\iota} & A \end{array}$$

commutes. Furthermore, by the uniqueness part of the statement of the universal property of $\mathrm{Bl}_Z A$, we deduce easily that $\tilde{\iota}$ is also an involution.

Now the blow-up $\mathrm{Bl}_Z A$ is a projective variety over k with an action of $\mathbf{Z}/2\mathbf{Z}$ via the involution $\tilde{\iota}$. Therefore, the *categorical quotient* $X := \mathrm{Bl}_Z A/(\mathbf{Z}/2\mathbf{Z})$ exists in the category of schemes. The scheme X constructed in this way is called the *Kummer surface associated to* A, and turns out to be a K3 surface. In particular, $H^1(X, \mathcal{O}_X) = 0$, so the Picard scheme $\mathrm{Pic}_{X/k}$ of X is étale, and $\mathrm{Pic}(X)$ is therefore a finitely-generated abelian group.

For an arbitrary proper scheme Y/k, recall that the Néron–Severi group of Y is a finitely generated abelian group [SGA6, Exposé XIII, Théorème 5.1]. Therefore, we may consider the *Picard number* $\rho(Y)$, defined as the rank of the Néron–Severi group of Y (which for X is just the rank of the Picard group). It is proven in [Shi79, Proposition 3.1] that the Picard number of X is given by the formula $\rho(X) = 16 + \rho(A)$. However, a crucial step in Shioda's proof relies on a calculation in [Shi75], of which we are not able to access a copy online. In this note, we give an explicit proof of this fact that is entirely self-contained. We do not use any Hodge theory, e.g. we do not study the complement of $\mathrm{Pic}(X)$ in $H^2(X, \mathbf{Z})$, i.e. the transcendental lattice $T(X)$.

Theorem 1.1. *Let k be a separably closed field of characteristic not 2, and A/k an abelian surface. Let X denote the Kummer surface associated to A. Then the Picard number of X is given by*

$$\rho(X) = 16 + \rho(A).$$

2. Preliminaries

In this section, we record a crucial result about abelian varieties that we will need. Let k be a separably closed field of characteristic not 2, and let A/k be an abelian variety (of arbitrary dimension). The group $\mathbf{Z}/2\mathbf{Z}$ acts on $\mathrm{Pic}(A)$ by $\mathcal{L} \mapsto [-1]^*\mathcal{L}$, and this action descends to one on the subgroup of numerically trivial line bundles $\mathrm{Pic}^0(A)$. In particular, the sequence

$$0 \to \mathrm{Pic}^0(A) \to \mathrm{Pic}(A) \to \mathrm{NS}(A) \to 0 \tag{1}$$

is an exact sequence of $\mathbf{Z}/2\mathbf{Z}$-modules.

Proposition 2.1. *Taking $\mathbf{Z}/2\mathbf{Z}$-invariants in* (1)*, we obtain an exact sequence*

$$0 \to \widehat{A}[2](k) \to \mathrm{Pic}(A)^{\mathbf{Z}/2\mathbf{Z}} \to \mathrm{NS}(A) \to 0,$$

where $\widehat{A}$ is the dual abelian variety of A. In particular, $\mathrm{Pic}(A)^{\mathbf{Z}/2\mathbf{Z}}$ *is a finitely generated abelian group of rank equal to the Picard number $\rho(A)$ of A.*

Proof. Let us first recall several facts about multiplication by n on A:
(a) For $\mathcal{L} \in \mathrm{Pic}^0(A)$, $[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n}$ [Con15, Proof of Lemma 5.2.5].
(b) For $\mathcal{L} \in \mathrm{Pic}(A)$, we have

$$[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n^2} \mod \mathrm{Pic}^0(A).$$

In other words, $[n]^*$ has the effect of multiplication by n^2 on $\mathrm{NS}(A)$ [Con15, Lemma 7.5.2].

Granting these facts, let us first compute $\mathrm{Pic}^0(A)^{\mathbf{Z}/2\mathbf{Z}}$. By (a), a line bundle $\mathcal{L} \in \mathrm{Pic}^0(A)$ satisfies $[-1]^*\mathcal{L} = \mathcal{L}$ precisely when $\mathcal{L}^{\otimes 2} \simeq 0$. In other words,

$$\mathrm{Pic}^0(A)^{\mathbf{Z}/2\mathbf{Z}} = \widehat{A}[2](k).$$

Next, by (b) above, the $\mathbf{Z}/2\mathbf{Z}$-action on $\mathrm{NS}(A)$ is trivial, so $\mathrm{NS}(A)^{\mathbf{Z}/2\mathbf{Z}} = \mathrm{NS}(A)$. Finally, we show that $H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{Pic}^0(A)) = 0$, which will complete the proof of the proposition. Write σ for the generator of $\mathbf{Z}/2\mathbf{Z}$, and let N denote the "norm" map

$$\begin{aligned} N: \quad \mathrm{Pic}^0(A) &\to \mathrm{Pic}^0(A) \\ \mathcal{L} &\mapsto \mathcal{L} \otimes [-1]^*\mathcal{L}. \end{aligned}$$

By the calculation of the cohomology of finite cyclic groups,

$$H^1(\mathbf{Z}/2\mathbf{Z}, \mathrm{Pic}^0(A)) \simeq \ker N/(\sigma - 1)\,\mathrm{Pic}^0(A).$$

By (a) above, we have $\ker N = \mathrm{Pic}^0(A)$. On the other hand, for $\mathcal{L} \in \mathrm{Pic}^0(A)$,

$$(\sigma - 1)(\mathcal{L}) = [-1]^*\mathcal{L} \otimes \mathcal{L}^\vee \simeq (\mathcal{L}^\vee)^{\otimes 2}.$$

In other words, $(\sigma-1)$ has the effect of multiplication by -2 on $\mathrm{Pic}^0(A)$. But now recall that $\mathrm{Pic}^0(A) = \widehat{A}(k)$, and $[-2] : \widehat{A} \to \widehat{A}$ is surjective étale. Since k is separably closed, the map on k-points is surjective, and therefore $(\sigma - 1)\,\mathrm{Pic}^0(A) = \mathrm{Pic}^0(A)$, from which the vanishing of the cohomology group in question follows. □

3. Proof of Theorem 1.1

Let $p_1, \ldots, p_{16}$ be the points in $A[2](k)$, let E_i be the preimage of p_i in $\mathrm{Bl}_Z A$ (the exceptional divisors), and let $\widetilde{E}_i$ denote the image of E_i in the quotient X. Define

$$\begin{aligned} \widetilde{E} &:= \bigcup_{i=1}^{16} \widetilde{E}_i. \\ E &:= \bigcup_{i=1}^{16} E_i. \end{aligned}$$

It is proven in [Ba\u{0}1, Theorem 10.6] that the $\widetilde{E}_i$'s are irreducible divisors in X, and are furthermore $\mathbf{Z}$-linearly independent in $\mathrm{Pic}(X)$. Therefore, identifying the Weil class group of X with its Picard group (by smoothness), we obtain the exact sequence

$$0 \to \bigoplus_{i=1}^{16} \mathbf{Z}[\widetilde{E}_i] \to \mathrm{Pic}(X) \to \mathrm{Pic}(X - \widetilde{E}) \to 0 \tag{2}$$

with the group on the left isomorphic in the obvious way to $\mathbf{Z}^{16}$. Now the formation of the quotient $\pi : \mathrm{Bl}_Z A \to X$ commutes with open immersions. More precisely, for any open subscheme $\widetilde{V} \subset X$, if we let $V := \pi^{-1}(\widetilde{V})$, then the map $\pi\colon V \to \widetilde{V}$ is the categorical quotient of V by $\mathbf{Z}/2\mathbf{Z}$. Therefore,

$$X - \widetilde{E} \simeq \pi^{-1}(X - \widetilde{E})/(\mathbf{Z}/2\mathbf{Z}).$$

But now observe that

$$\pi^{-1}(X - \widetilde{E}) \simeq \mathrm{Bl}_Z A - E \simeq A - A[2],$$

so

$$(A - A[2])/(\mathbf{Z}/2\mathbf{Z}) \simeq X - \widetilde{E}.$$

Therefore, the result $\rho(X) = 16 + \rho(A)$ will follow from (2) *if* we can show that $\mathrm{Pic}((A - A[2])/(\mathbf{Z}/2\mathbf{Z}))$ is a finitely generated abelian group of rank equal to $\rho(A)$. To this end, define $U \coloneqq A - A[2]$ and $G \coloneqq \mathbf{Z}/2\mathbf{Z}$. Since the action of G on U is free, the quotient map $U \to U/G$ is a *Galois cover* [SGA3, Exposé V, Théorème 4.1(iii) and (iv)], and we have an associated Hochschild-Serre spectral sequence

$$H^i(G, H^j(U, \mathbf{G}_m)) \implies H^{i+j}(U/G, \mathbf{G}_m).$$

The low-degree terms of this spectral sequence are

$$0 \to H^1(G, H^0(U, \mathbf{G}_m)) \to \mathrm{Pic}(U/G) \to \mathrm{Pic}(U)^G \to H^2(G, H^0(U, \mathbf{G}_m)).$$

Now before we calculate any cohomology, we make the observation that

$$H^0(U, \mathbf{G}_m) = H^0(A, \mathbf{G}_m) = k^\times.$$

Indeed, this is true by Hartogs' Lemma since A is smooth (a fortiori normal!) and $A \setminus U$ is codimension 2 in A. Also, observe that the Galois action of G on $k^\times$ is trivial.

We now compute $H^1(G, H^0(U, \mathbf{G}_m))$. By the discussion above, this is isomorphic to $\mathrm{Hom}(G, k^\times) = \mu_2(k)$. On the other hand, by the calculation of the cohomology of finite cyclic groups, $H^2(G, k^\times) \simeq k^\times/(k^\times)^2$. Since k is separably closed, this is zero and so $\mathrm{Pic}(U/G)$ sits in an exact sequence

$$0 \to \mu_2(k) \to \mathrm{Pic}(U/G) \to \mathrm{Pic}(U)^G \to 0. \tag{3}$$

By the equivalence of the Picard group with the Weil divisor class group for regular schemes, and because $A \backslash U$ has codimension 2 in A, $\mathrm{Pic}(U) = \mathrm{Pic}(A)$. By Proposition 2.1, $\mathrm{rk}\,\mathrm{Pic}(A)^G = \rho(A)$. Combining this with (3) yields the equality

$$\mathrm{rk}\,\mathrm{Pic}((A - A[2])/(\mathbf{Z}/2\mathbf{Z})) = \rho(A),$$

as desired.

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